TP06 — Inflation and string theory

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Abstract

We briefly review the inflationary paradigm for resolving outstanding issues with standard cosmology. The standard model cannot account for inflation, and so we consider string theory models which offer some hope. In particular, we examine scalar fields arising from type IIB compactifications and in doing so provide an overview of string theory.

1 Introduction

The standard cosmological model, based on observations of homogeneity and isotropy, has several shortcomings [1]. There is the *horizon problem*: if parts of the universe have never been in causal contact, why are they correlated? The *flatness problem*: the universe is measured to be exceedingly flat, which itself requires a fine-tuning of the cosmological parameters during the early universe. The *monopole problem*: there are particles which have small annihilation cross sections, and we expect these to have been produced as the universe cooled. Why do we not observe any today?

Inflation has been proposed to explain this disparity between what we expect and what we see [2]. A period of rapid expansion of the universe at an early stage would ensure that the observable universe today was previously an exceedingly small part of the universe. Since that small part was inside the particle horizon at that time, we can explain the horizon problem. Furthermore, the expansion reduces the density of monopoles and sends the curvature of the universe near to unity.

We suppose that there is a hypothetical scalar field ϕ, called the inflaton, with an associated potential $V(\phi)$. For inflation to occur so that we can resolve the observational problems, we will see that we require the potential to have a local minimum and a section where it is almost flat. Furthermore, we require a certain scale of expansion. We can check if a given potential satisfies these conditions, and so build a suitable model for the inflaton. The only fundamental scalar field in the standard model, the Higgs field, is problematic, and we need to look beyond the standard model to identify the inflaton [2].

Supersymmetry is the idea that we can extend the usual Poincaré symmetry algebra of the standard model with N additional fermionic generators which change bosons into fermions and vice versa. This approach can be developed methodically to give rise to further particle content in the theory. Since we do not observe supersymmetric particles at low energies, a realistic model will have supersymmetry breaking that leads to the observed standard model particles and the standard model gauge group $SU(3)\otimes SU(2)\otimes U(1)$. Since local Poincaré symmetry gives rise to general relativity, it is natural to consider local supersymmetry. This gives rise to a theory called supergravity.

Superstring theory (or string theory) proposes small strings whose different vibrational modes at low energies account for the different particles that we observe today [5]. String theory can describe open or closed strings that have different boundary conditions (e.g. open strings attached at both ends to higher-dimensional p-branes, open strings with one end or both ends free and closed strings), which obey different symmetries. These various considerations lead to different types of string theory,[1] but we will consider here the so-called Type IIB theory. For consistency, the theory requires 10 dimensions – we observe 4 of them but the other 6 become part of the internal space, M.

There are various models for the structure of the internal space. Each model will provide us with a particular form of the effective theory which will be a supergravity action in 4 dimensions. We can then proceed to see if any of the scalar fields in the effective theory could be the inflaton field. Our method of approach will be to examine the field's potential for a minima and for the existence of a suitably flat direction. We could also check to see if the field's potential predicts the same distribution of the cosmic microwave background (CMB) and other observational data.

As the authors show in [7], the effective potential due to the Kähler moduli[2] satisfies the necessary constraints for inflation. However, they employ a method whereby they fix all the other fields appearing in the theory. This approach could be problematic, and we will consider a more general potential that includes two more fields that appear in the effective theory (called the dilaton) along with the Kähler moduli.

We will briefly review the Friedmann-Robertson-Walker cosmology, examine the horizon problem and discuss how inflation can solve these problems. We will then discuss some details about inflationary models, and see how we can test the models by comparing with observational data. We then present some details of string theory and look at the various scalar fields that arise from the effective IIB theory. We then compute the effective potentials arising from the various scalar fields. In particular, we review the work of [7] (which considers only the Kähler moduli), and then extend it by considering the dilaton too.

[1] However, all 5 realistic string theories are related by various dualities to a unique theory called M-theory [3].

[2] Kähler moduli are a kind of scalar field that appear in the effective theory.

2 Inflationary cosmology

2.1 Cosmology

When discussing cosmology, it helps to define comoving observers. They are at rest with respect to the CMB, i.e. they observe no dipole moment. We can also assign a set of coordinates for these observers: $(t = \tau, \vec{x})$, where τ is their proper time and $\vec{x}$ is a constant 3-vector.

The observation that the universe is isotropic and homogeneous puts severe constraints on the form of the metric. In the above coordinates, we express such a metric as the so-called FRW metric:

$$ds^2 = dt^2 - a^2(t) \times \left[\frac{dr^2}{1 - kr^2} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \right]$$

where the components of the Riemann tensor are $R_{ijkl} = K(g_{ik}g_{jl} - g_{il}g_{jk})$ and $k = K/|K|$ specifies k.[3] If $K = 0$ (i.e. the 3-space is flat), then $k = 0$. $a(t)$ acts as a scale factor.

To make the discussion less cluttered, we expand $a(t)$ about the current time t_0:

$$\begin{aligned} a(t) &= a(t_0 - (t_0 - t)) \\ &= a(t_0 - \delta t) \\ &= a(t_0) - \dot{a}(t_0)\delta t + ... \\ &= a(t_0)[1 - H(t_0)\delta t ...] \end{aligned}$$

and define $H(t) \equiv \dot{a}(t)/a(t)$, the Hubble parameter.

We consider solutions to the Einstein equation $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\kappa T_{\mu\nu}$ in the presence of a perfect fluid, with the constraints of homogeneity and isotropy discussed above.[4] The

[3] We use Latin indices i, j, k, l to range over the space indices $1, 2, 3$ here only.

[4] We use Greek indices μ, ν, ρ, σ to range over the spacetime indices $0, 1, 2, 3$.

stress-energy tensor of the fluid is given by

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu - pg^{\mu\nu} \tag{1}$$

where u is the 4-velocity of a particle in the fluid and $\rho = \rho(t)$, $p = p(t)$ are the density and pressure, respectively. These are functions of time only.

We rewrite the Einstein equations as $R_{\mu\nu} = -\kappa(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu})$ by using the trace of the Einstein equation and then we solve it for this version of the FRW metric.

After some calculation, we see that all the off-diagonal components of R_{ab} are zero, and that the equations for the space indices ($R_{ii} = -\kappa(T_{ii} - \frac{1}{2}Tg_{ii})$) are all linearly dependent. This gives us two independent equations, called the Friedmann equations [1]:

$$\begin{aligned} \ddot{a} &= -\frac{1}{6}\kappa(\rho + 3p)a \\ \dot{a}^2 &= \frac{1}{3}\kappa\rho a^2 - k \end{aligned}$$

In the above coordinates, geodesics trace out lines of constant θ and ϕ. We can obtain an explicit form for $a(t)$ by solving the Friedmann equations once we have an equation of state (an equation relating p and ρ). Some usual models that are considered include a matter dominated and a radiation dominated universe. In usual models, we find that the universe is continuously decelerating in its expansion, i.e. $\ddot{a} < 0$. In this case, the particle horizon (the region from which a light, and therefore information, signal could have reached that point) is a finite quantity.

Experimentally, however, we see the CMB, which is uniform to 1 part in 100,000. This is known as the *horizon problem*: if parts of the universe have never been in causal contact, why are they correlated? We might expect that objects in the universe were nearer at earlier times, however working through the equations we find that if we go to a cosmic time when objects get closer by a factor of z, the horizon shrinks by a factor of $3z^{-3/2}$ for a matter dominated universe and $2z^{-2}$ for a radiation dominated universe.

2.2 Scalar fields and inflation

The Lagrangian density of a scalar field is given by[5]

$$\mathcal{L} = \sqrt{-g}\left[\tfrac{1}{2}g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - V(\phi)\right] \tag{2}$$

and this induces the action $S = \int \mathcal{L} d^4x$. We can obtain the stress energy tensor of the field by varying the action with respect to the metric. We obtain

$$T_{\mu\nu} = \partial_\mu\phi\,\partial_\nu\phi - g_{\mu\nu}\left[\tfrac{1}{2}\partial_\rho\phi\,\partial^\rho\phi - V(\phi)\right] \tag{3}$$

If we compare equations (1) and (3) in comoving coordinates and ignore spatial variations (because of homogeneity), we can make the identification

$$\begin{aligned} \rho_\phi &= \tfrac{1}{2}\dot{\phi}^2 + V(\phi) \\ p_\phi &= \tfrac{1}{2}\dot{\phi}^2 - V(\phi) \end{aligned}$$

To calculate the equation of motion in an FRW cosmology, we look at the continuity equation $\nabla_\mu T^{\mu\nu} = 0$ (and ignore spatial variations) to obtain

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0 \tag{4}$$

This equation would look like the motion of a particle in a Newtonian gravitational field were it not for the friction-like term $3H\dot{\phi}$. As the field rolls down its potential, it will oscillate about a local minimum (if there is one), and the oscillations will dampen out due to the friction term.

If we ignore the spatial curvature in the Friedmann equations, we get an expression for the Hubble parameter

$$H^2 = \tfrac{1}{3}\left[\tfrac{1}{2}\dot{\phi}^2 + V(\phi)\right] \tag{5}$$

The condition for inflation, $\ddot{a} > 0$, is equivalent to $\dot{\phi}^2 < V(\phi)$ (this can also be obtained from the Friedmann equations). If we consider a situation where $\dot{\phi}^2 \ll V(\phi) \Rightarrow \ddot{\phi} \ll V'(\phi)$ (the slow-roll approximation), we can simplify our equations (4), (5) considerably [1]:

$$\begin{aligned} V'(\phi) &= -3H\dot{\phi} \\ V(\phi) &= 3H^2 \end{aligned}$$

[5] Repeated indices are summed over.

In particular, the last equation is simply a first order differential equation, and can be solved to give

$$a(t) \propto \exp\sqrt{\tfrac{1}{3}V(\phi)}t$$

Thus the universe grows exponentially for as long as the slow-roll condition holds. We can also cast the slow roll condition in terms of some dimensionless quantities [2]:

$$\epsilon \equiv \frac{1}{2}\left(\frac{V'}{V}\right)^2 \ll 1 \quad (6)$$

$$\eta \equiv \frac{V''}{V} \ll 1 \quad (7)$$

$$\xi \equiv \frac{V'V'''}{V^2} \ll \epsilon \quad (8)$$

The universe is not currently in an inflationary phase, so it is necessary that inflation stops. As noted above, this requires the existence of a local minima.

We can calculate the total amount of expansion by integrating the Hubble parameter for as long as the scalar field is in the slow roll regime [1]

$$\begin{aligned} N &= \int_{t_0}^{t_1} H dt = \int_{\phi_0}^{\phi_1} H\dot{\phi}^{-1}\dot{\phi}\ dt \\ &= \int_{\phi_0}^{\phi_1} H\dot{\phi}^{-1}\ d\phi = -\int_{\phi_0}^{\phi_1} \frac{H}{\dot{\phi}}\frac{V\dot{\phi}}{HV'}d\phi \\ &= -\int_{\phi_0}^{\phi_1} \frac{V}{V'}d\phi \end{aligned}$$

with the penultimate step enabled by noting $V\dot{\phi}/HV' = -1$ (this is valid in the slow-roll regime). N measures the number of e-foldings, i.e. it represents an increase by a scale factor e^N. This can be seen from the fact that $H = \dot{a}/a$. From observational considerations, we require $N \simeq 60 - 70$ [1].

We define the spectral index, $n(k)$, by $n(k) - 1 \equiv d\log\mathcal{P}_\mathcal{R}/d\log k$, where $\mathcal{P}_\mathcal{R}(k)$ is the power spectrum of the curvature perturbations[6]. Once we have a form of the potential, we can compute the slow roll parameters, and then compute the spectral index [2] by solving

$$\begin{aligned} n - 1 &= 2\eta - 6\epsilon \\ \frac{dn}{d\log k} &= 24\epsilon^2 - 16\epsilon\eta + 2\xi^2 \end{aligned}$$

We can then compare the computed spectral index with what we observe to further test our model.

These can provide very precise constraints on the model, but the general feature we are looking for is a local minimum and the existence of a flat section in the potential.

3 Effective string theory and supergravity

3.1 Superstring theory

In the standard model we consider particles which trace out worldlines $X^\mu(\tau)$ that exist in 4-dimensional spacetime. These lines do not change if one changes the parametrization (i.e. perform an affine transformation on the parameter), something called reparametrization invariance. The action for a particle is proportional to the length of its worldline. This action obeys Poincaré invariance and the local $SU(3) \otimes SU(2) \otimes U(1)$ gauge invariance gives rise to the interaction terms. In quantum field theory, we calculate the probability that a certain set of incoming particles with given momenta will leave as certain set of other particles with a specified momenta.

In IIB string theory, the closed strings trace out worldsheets as they move in the time direction in 10-dimensional spacetime. The action in string theory is the natural generalization from particles, and is proportional to the area of the worldsheet. In string theory, the spectrum of string vibration modes gives rise to all the particles we observe at low energies, and so we do not need interaction terms in the action. We simply consider the free theory of massless bosons and Majorana fermions[7].

[6] To be precise, it is defined as the total variance of the perturbations per unit logarithmic interval in k [1].

[7] A Majorana fermion is one which remains unchanged under charge conjugation. It has the same number of degrees of freedom as a Weyl spinor i.e. just

We consider bosonic and fermionic functions $X^\mu(\sigma,\tau)$ and $\psi^\mu(\sigma,\tau)$ of the two coordinates of the worldsheet. The action is constructed to be invariant under local supersymmetry (this automatically includes local Poincaré symmetry).[8] The action is also invariant under reparametrizations of the worldsheet coordinates (σ,τ) and rescalings of the worldsheet metric (called Weyl transformations).

The theory contains states with negative norm which are in principle problematic[9], but these can be shown to be unphysical and can be removed by the so-called super-Virasoro constraints [3]. This, however, only works in 10 spacetime dimensions and this is why we need the extra dimensions in the theory.

3.2 Low energy effective theory

At large distances compared to the size of the strings (or at low energies), the theory should reduce to an effective field theory in four spacetime dimensions, since that is the world we observe. This gives rise to 4-dimensional supergravity coupled with matter. In particular, for $N = 1$ supergravity, the bosonic part of the action [6] looks like[10]

$$\begin{aligned} \mathcal{L}_B &= \sqrt{-g}\left[-\tfrac{1}{2}R + K_{a\bar{b}}\partial_\mu\phi^a\partial^\mu\bar{\phi}^{\bar{b}}\right] \\ &\quad - V(\phi^a,\bar{\phi}^{\bar{a}}) \qquad (9) \\ V(\phi^a,\bar{\phi}^{\bar{a}}) &= e^K\left[K^{a\bar{b}}F_a\bar{F}_{\bar{b}} - 3|W|^2\right] \qquad (10) \end{aligned}$$

where

$$\begin{aligned} F_a &\equiv W(\partial_a K) + \partial_a W \\ K_{a\bar{b}} &\equiv \partial_a\partial_{\bar{b}}K \\ \delta^a_c &\equiv K^{a\bar{b}}K_{\bar{b}c} \end{aligned}$$

The partial derivatives are with respect to the complex scalar fields appearing in the theory, and a derivative with respect to a barred index implies differentiation with respect to the conjugate of the field. $K = K(\phi^a,\bar{\phi}^{\bar{a}})$ is the Kähler potential, and $W = W(\phi^a)$ is the superpotential. These two functions are enough to completely specify the 4D supergravity. $K_{a\bar{b}}$ here acts a like an analogue of the ordinary metric of spacetime.

We obtain the stress energy tensor by varying the action with respect to the metric, and the equation of motion by varying of the action with respect to the fields and setting that to zero:

$$\begin{aligned} T_{\rho\sigma} &= K_{a\bar{b}}\partial_\rho\phi^a\partial_\sigma\bar{\phi}^{\bar{b}} - g_{\rho\sigma} \\ &\quad \times\left(\tfrac{1}{2}K_{a\bar{b}}\partial_\mu\phi^a\partial^\mu\bar{\phi}^{\bar{b}} - V\right) \\ 0 &= \Box\phi^d + 2\Gamma^d_{ab}\partial_\mu\phi^a\partial^\mu\phi^b + K^{\bar{c}d}\partial_{\bar{c}}V \end{aligned}$$

where $\Gamma^d_{ab} \equiv \frac{1}{2}K^{\bar{c}d}\partial_b K_{\bar{c}a}$. A derivation can be found in Appendix A.

Apart from the addition of the Ricci scalar, R, for the curved spacetime in (9), this is almost the same as the Lagrangian (2). To identify the two, we ought to bring the scalar fields into canonically normalized form. This can be done by introducing a new variable ϕ_c for each ϕ, where the two are related by [4]

$$\frac{1}{2}(d\phi^a_c)^2 = K_{a\bar{a}}\phi^a\bar{\phi}^{\bar{a}} \qquad (11)$$

since we have an overall factor of a half that didn't multiply the kinetic term in our 4D supergravity Lagrangian but a factor of $K_{a\bar{b}}$ that did. This is important because we derived all the relations regarding inflation when considering scalar fields with Lagrangians of the form (2). If we want to then apply these relations to scalar fields with Lagrangians of the form (9), then we ought to convert the latter type to the former.

There are various ways the extra dimensions can be compactified, and this will specify the precise form of $K = K(\phi^a,\bar{\phi}^{\bar{a}})$ and $W = W(\phi^a)$, as well as the various scalar fields that will arise (we have denoted all of these collectively as ϕ above).

In type IIB, the Kähler potential is the sum $K = K(S) + K(Z) + K(T)$, which at tree-level

one of the two chiral spinors that make up a Dirac spinor.

[8] In the conformal gauge, the worldsheet appears only globally supersymmetric, but the more fundamental formulation considers local supersymmetry. Furthermore, this worldsheet supersymmetry is equivalent to spacetime supersymmetry in 10 dimensions [3].

[9] since $\langle\psi|\psi\rangle > 0$ should always hold.

[10] We use Latin indices a, b, c to run over all the *complex* scalar fields appearing in the effective theory.

are

$$\begin{aligned} K(S) &= -\log(S+\bar{S}) \\ K(Z) &= -\log(-i\int_M \Omega\wedge\bar{\Omega}) \\ K(T) &= -2\log\mathcal{V}(T_i,\bar{T}_i) \end{aligned}$$

These terms are, respectively, the dilaton, the complex structure and the volume term.

S here is a complex scalar $S \equiv s + i\sigma$. The volume of the internal space M, given by $\mathcal{V}$, will actually depend only on the Kähler moduli, τ_i, but the indices in equations (9–10) refer to differentiation with respect to the complex scalar $T_i = \tau_i + ib_i$, and so we view $\mathcal{V}$ as a function of T_i for now. The integration in the K_{cs} term happens over M.

The superpotential is given by

$$W = \int_M G_3 \wedge \Omega$$

What these variables mean are largely irrelevant for the discussion here. The key point is that it does not depend on the Kähler moduli.

4 A toy model: the dilaton

To illustrate the machinery we have developed, we will first look at a very simple model. We consider the dilaton, $S \equiv s + i\sigma$, just by itself. The full Kähler potential is then just $K = -\log(S+\bar{S})$, and we consider a superpotential only linear in S: $W = a + bS$, where $a, b \in \mathbb{C}$. We obtain

$$\begin{aligned} K^{S\bar{S}} &= (S+\bar{S})^{-2} = (2s)^{-2} \\ F_S &= (b - (2s)^{-1}(a+bS)) \end{aligned}$$

Putting that into (10), we obtain for the potential

$$\begin{aligned} V = -\tfrac{1}{s}[&|a|^2 + |b|^2(s^2+\sigma^2) \\ &+ 4s\mathfrak{Re}\{a\bar{b}\} + 2\sigma\mathfrak{Im}\{a\bar{b}\}] \end{aligned}$$

Setting the first derivatives to zero gives us the extrema

$$\begin{aligned} \sigma_0 &= -\mathfrak{Im}\{a\bar{b}\}/|b|^2 \\ s_0 &= \pm\sqrt{|a|^2/|b|^2 - \sigma_0^2} \end{aligned}$$

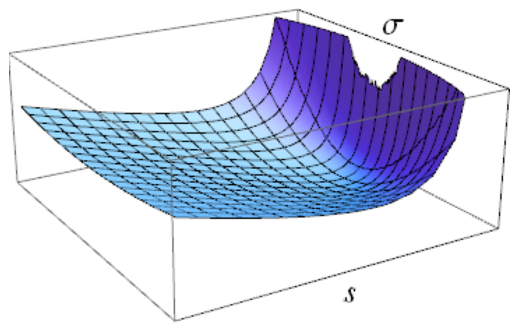

Figure 1: Minimum for the toy model, with $a = 10(1+i), b = 3+4i$.

In order for s_0 to be real, we require $|a|^2 > |b|^2\sigma_0^2$. To determine the nature of the extrema, we calculate the Hessian matrix $\partial^2 V/\partial x_i \partial x_j$ and evaluate it at each the extremum. If it is positive definite, then the extremum is a minimum. We find

$$\begin{aligned} \frac{\partial^2 V}{\partial\sigma^2} &= -\frac{2|b|^2}{s_0} \\ \frac{\partial^2 V}{\partial s^2} &= -\frac{2}{s_0^3}\left(|a|^2 - \frac{\mathfrak{Im}\{a\bar{b}\}^2}{|b|^2}\right) \\ \frac{\partial^2 V}{\partial s\partial\sigma} &= 0 \end{aligned}$$

We want this matrix to be positive definite, or equivalently, have all positive eigenvalues. Since this matrix is already in diagonal form, we simply check the diagonal elements. Since both terms are negative and s_0 enters as an odd power, we can choose the negative solution for s_0 to ensure this extremum is a minimum.

So we have a minimum located at

$$\begin{aligned} \sigma_0 &= -\mathfrak{Im}\{a\bar{b}\}/|b|^2 \\ s_0 &= -\sqrt{|a|^2/|b|^2 - \sigma_0^2} \end{aligned}$$

given that $|a|^2 > |b|^2\sigma^2$. The minimum is shown in Fig. 1.

We note that there is a slow-roll direction, if initially $s \ll 0$. We could compute the slow-roll parameters for this model, the number of e-foldings, the spectral index etc., but we will not do that here.

5 Kähler moduli

As described above, the Kähler moduli enter into the Kähler potential via the volume $\mathcal{V}$. We will consider the potential due to Kähler moduli only. We will largely follow the derivations presented in [7].

First, we consider the case where the potential is a 'no-scale' potential, i.e. it satisfies $K^{T_i\bar{T}_j}\partial_{T_i}K\partial_{\bar{T}_j}K = 3$. An example would be [7]:

$$\mathcal{V} = \alpha\left(\tau_i^{3/2} - \sum_{i=2}^{n}\lambda_i\tau_i^{3/2}\right)$$

where the λ_i and α are dependent on the specific model.

Since the superpotential does not depend on the moduli fields, we can simplify the expression for the potential:

$$\begin{aligned}V = e^K[&K^{T_i\bar{T}_j}(W\partial_{T_i}K)(\bar{W}\partial_{\bar{T}_j}K) - 3|W|^2 \\ &+ K^{c\bar{d}}(W\partial_c K + \partial_c W)(\bar{W}\partial_{\bar{d}}K + \partial_{\bar{d}}\bar{W})]\end{aligned}$$

Here, the c, d indices run over the complex structure and the dilaton only. We use the no-scale property to cancel the first two terms leaving us with

$$V = e^K[K^{c\bar{d}}(W\partial_c K + \partial_c W)(\bar{W}\partial_{\bar{d}}K + \partial_{\bar{d}}\bar{W})]$$

Since the no-scale potential is positive definite [8], locating the extrema is straightforward. We find them by solving

$$W\partial_c K + \partial_c W = 0$$

where, as before, c runs over the complex structure moduli and dilaton only. We denote the solution by W_0. In principle this fixes these fields and only leaves the Kähler moduli free; however this prescription may be problematic and we will test it in a more thorough analysis. Indeed, that will be the primary goal of this investigation. For the remainder of this section, we will consider the dilaton and complex structure fixed, and so the indices will only run over the moduli fields.

So for a no-scale potential (with the dilaton and complex structure fixed), the potential is zero. We now add non-perturbative effects to the superpotential [7], and these do depend on the Kähler moduli:

$$W = W_0 + \sum_{i=2}^{n} A_i e^{-a_i T_i}$$

where A_i is model dependent and $a_i = 2\pi/N,\ N \in \mathbb{Z}_+$.

We also include α' corrections in the volume term as per [9] to get:

$$\mathcal{V} = \alpha\left(\tau_i^{3/2} - \sum_{i=2}^{n}\lambda_i\tau_i^{3/2} + \frac{\xi}{2}\right) \qquad (12)$$

where $\xi = -\chi(M)/2(2\pi)^3$, and $\chi(M)$ is the Euler characteristic of the space.

Since W is no longer independent of the Kähler moduli, the derivatives with respect to it no longer disappear. We get for the potential

$$\begin{aligned}V &= e^K K^{T_i\bar{T}_j}\Big[\partial_{T_i}K\partial_{\bar{T}_j}K|W|^2 - 3|W|^2 \\ &\quad + \partial_{T_i}W\partial_{\bar{T}_j}\bar{W} + (W\partial_{T_i}K\partial_{\bar{T}_j}\bar{W} + c.c.)\Big] \\ &= e^K K^{T_i\bar{T}_j}\Big[\partial_{T_i}W\partial_{\bar{T}_j}\bar{W} \\ &\quad + (W\partial_{T_i}K\partial_{\bar{T}_j}\bar{W} + c.c.)\Big]\end{aligned}$$

We present the full derivation of the potential in Appendix B. The final form of the potential is

$$\begin{aligned}V &= \frac{8}{3\mathcal{V}\alpha}\sum_{i=2}^{n}\frac{(a_iA_ie^{-a_i\tau_i})^2\sqrt{\tau_i}}{\lambda_i} \\ &\quad - \frac{4W_0}{\mathcal{V}^2}\sum_{i=2}^{n}a_iA_i\tau_ie^{-a_i\tau_i} + \frac{3\xi W_0^2}{4\mathcal{V}^3}\end{aligned}$$

Differentiating with respect to τ_i at constant $\mathcal{V}$ as in [8] and equating to 0 to locate the extrema, we get

$$\begin{aligned}a_iA_ie^{-a_i\tau_i} &= \frac{3\alpha\lambda_i\sqrt{\tau_i}W_0}{\mathcal{V}}\frac{1-a_i\tau_i}{1-4a_i\tau_i} \\ &\simeq \frac{3\alpha\lambda_i\sqrt{\tau_i}W_0}{4\mathcal{V}}\end{aligned}$$

where we make the approximation that $a_i\tau_i \gg 1$ as in [7].

We then calculate the Hessian matrix, $\partial^2 V/\partial\tau_i\partial\tau_j$, and evaluate it at the extrema to determine the nature of the extrema as before. The Hessian is diagonal since $\partial V/\partial\tau_i$ only contains τ_i and we get

$$\frac{\partial^2 V}{\partial\tau_i\partial\tau_j} = \frac{3\alpha\lambda_i W_0^2}{\mathcal{V}^3\sqrt{\tau_i}}\left(a_i^2\tau_i^2 + a_i\tau_i - \frac{1}{8}\right)\delta_{ij}$$

We can rewrite the quadratic as $(a_i\tau_i + \frac{1}{2})^2 - \frac{3}{8}$ and as we are considering the limit $\mathcal{V} \to \infty$, $a_i\tau_i \to \log\mathcal{V}$ as in [8], we can confirm that a minimum exists as long as $\alpha\lambda_i > 0 \; \forall \; i$.

6 Kähler moduli and dilaton

We now consider not fixing the dilaton and combining the potentials from the previous two sections. The potentials are

$$\begin{aligned} K &= -2\log\mathcal{V} - \log(S + \bar{S}) \\ W &= W_0 + \sum_{i=2}^{n} A_i e^{-a_i T_i} + c + dS \end{aligned}$$

where W_0 is the superpotential minimized with respect to the complex structure only, and $\mathcal{V}$ is as in eqn. (12).

The form of the potential is[11]

$$\begin{aligned} V = &\frac{4}{3\mathcal{V}\alpha s}\sum_{i=2}^{n}\frac{(a_i A_i e^{-a_i\tau_i})^2\sqrt{\tau_i}}{\lambda_i} \\ &- \frac{2W}{\mathcal{V}^2 s}\sum_{i=2}^{n} a_i A_i \tau_i e^{-a_i\tau_i} \\ &+ \frac{2}{\mathcal{V}^2}\left(s|c|^2 - \mathfrak{Re}\{\bar{W}c\}\right) + \frac{|W|^2}{2\mathcal{V}^2 s}\left(1 + \frac{3\xi}{4\mathcal{V}}\right) \end{aligned}$$

To declutter the notation, we introduce $\gamma \equiv 1 + 3\xi/4\mathcal{V}$, and $f \equiv \mathfrak{Re}\{\bar{W}b\}$. Note that γ is a constant with respect to the fields but f depends on s through W. We introduce $\dot{W} \equiv \partial W/\partial s, W' \equiv \partial W/\partial\sigma$ and $\dot{f} \equiv \partial f/\partial s$ (all other derivatives of f are zero).

Differentiating with respect to each τ_i, s and σ and equating to zero gives a condition for the extrema. We get

$$\begin{aligned} \tau_i: &\quad a_i A_i e^{-a_i\tau_i} \simeq \frac{3\alpha\lambda_i\sqrt{\tau_i}W_0}{4\mathcal{V}} \\ s: &\quad \textstyle\sum_{i=2}^{n}\frac{(a_i A_i e^{-a_i\tau_i})^2\sqrt{\tau_i}}{\lambda_i} = \frac{3\alpha\gamma W^2}{8\mathcal{V}} \\ \sigma: &\quad \textstyle\sum_{i=2}^{n} a_i A_i \tau_i e^{-a_i\tau_i} = \gamma W \end{aligned}$$

We can then evaluate the Hessian matrix. Since i runs from 2 to n, and since we have the s, σ fields too, the matrix will be a $\mathrm{Dim}(n+1)$ symmetric matrix. The matrix will look like

$$\begin{pmatrix} \ddots & & & \vdots & \vdots \\ & \frac{\partial^2 V}{\partial\tau_i\partial\tau_j} & & \frac{\partial^2 V}{\partial\tau_i\partial s} & \frac{\partial^2 V}{\partial\tau_i\partial\sigma} \\ & & \ddots & \vdots & \vdots \\ \cdots & \frac{\partial^2 V}{\partial\tau_j\partial s} & \cdots & \frac{\partial^2 V}{\partial s^2} & \frac{\partial^2 V}{\partial\sigma\partial s} \\ \cdots & \frac{\partial^2 V}{\partial\tau_i\partial\sigma} & \cdots & \frac{\partial^2 V}{\partial\sigma\partial s} & \frac{\partial^2 V}{\partial\sigma^2} \end{pmatrix}$$

The second derivatives evaluated at the extrema are

$$\begin{aligned} \frac{\partial^2 V}{\partial\tau_i\partial\tau_j} &= \frac{3\alpha\lambda_i W^2}{2\mathcal{V}^3\sqrt{\tau_i}}\left(a_i^2\tau_i^2 + a_i\tau_i - \frac{1}{8}\right)\delta_{ij} \\ \frac{\partial^2 V}{\partial\tau_i\partial s} &= \frac{3\alpha\lambda_i\sqrt{\tau_i}W}{2s\mathcal{V}^3}\left(\frac{3W}{4s} + \dot{W}(a_i\tau_i - 1)\right) \\ \frac{\partial^2 V}{\partial\tau_i\partial\sigma} &= \frac{3\alpha\lambda_i\sqrt{\tau_i}W'W}{2s\mathcal{V}^3}(a_i\tau_i - 1) \\ \frac{\partial^2 V}{\partial s^2} &= \frac{\gamma(\dot{W})^2}{s\mathcal{V}^2} \\ \frac{\partial^2 V}{\partial\sigma\partial s} &= \frac{\gamma W'\dot{W}}{s\mathcal{V}^2} \\ \frac{\partial^2 V}{\partial\sigma^2} &= \frac{\gamma(W')^2}{s\mathcal{V}^2} \end{aligned}$$

For the extrema to be a minimum, we need all the eigenvalues to be positive. We notice that the upper left block is diagonal and is $\mathcal{O}(a_i^2\tau_i^{3/2}\mathcal{V}^{-3})$, the lower right block is $\mathcal{O}(\mathcal{V}^{-2})$ and the remaining elements are $\mathcal{O}(a_i\tau_i^{3/2}\mathcal{V}^{-3})$. Since we are taking the limit $\mathcal{V} \to \infty$, $a_i\tau_i \to \log\mathcal{V}$, we disregard the $\partial^2 V/\partial\tau_i\partial s, \partial^2 V/\partial\tau_i\partial\sigma$ terms as subleading.

We determine the eigenvalues of the lower right block by solving its characteristic equation

$$\begin{aligned} 0 &= \lambda\left[\lambda - \frac{\gamma}{s\mathcal{V}^2}\left((\dot{W})^2 + (W')^2\right)\right] \\ \Rightarrow \lambda &= 0, \frac{\gamma}{s\mathcal{V}^2}\left((\dot{W})^2 + (W')^2\right) \end{aligned}$$

[11] The full derivation of this is in Appendix C. The first and second derivatives of the potential, which are used later, can be found there too.

The appearance of a zero eigenvalue could be problematic as we may not be able to diagonalize the matrix. However if we actually can diagonalize the lower right block, then we can determine that

$$V(\vec{x}_0+\vec{\delta})-V(\vec{x}_0)=\frac{1}{2}\vec{\delta}^T\cdot H\cdot\vec{\delta}>0,\ \forall\ \vec{\delta}$$

(where $\vec{x}_0$ is a vector in the parameter space that is the extremum, i.e. $\vec{\nabla}V(\vec{x}_0)=0$) if the one eigenvalue is positive.[12]

What we have, is a matrix like

$$\begin{pmatrix} a_{11} & \sqrt{a_{11}a_{22}} \\ \sqrt{a_{11}a_{22}} & a_{22} \end{pmatrix}$$

with eigenvalues $\lambda=0,\ a_{11}+a_{22}$. It can be diagonalized by the normalized eigenvectors

$$\frac{1}{\sqrt{a_{11}+a_{22}}}\begin{pmatrix} \sqrt{a_{11}} \\ \sqrt{a_{22}} \end{pmatrix},\frac{1}{\sqrt{a_{11}+a_{22}}}\begin{pmatrix} \sqrt{a_{22}} \\ -\sqrt{a_{11}} \end{pmatrix}$$

to bring the matrix into the form

$$\begin{pmatrix} a_{11}+a_{22} & 0 \\ 0 & 0 \end{pmatrix}$$

The conditions for minimum then are the same as the ones from the previous section and one additional constraint from the above considerations:

$$\begin{aligned} \alpha\lambda_i &> 0 \\ \gamma/s &> 0 \end{aligned}$$

7 Discussion

We have succeeded then, in confirming the existence of minima, even if the dilaton remains unfixed, and if it enters as a linear function in the superpotential. We have also been able to obtain an additional constraint, namely $\gamma/s=(1+3\xi/4\mathcal{V})/s>0$. This will ensure that inflation can stop as the field settles into the minimum.

We can get a feel for the nature of these minima by plotting the potential. All the 'τ_i's enter into the function symmetrically, so there are four variables of interest really – a pair of τ_is, s and σ. We can examine the minima by fixing two of the variables at the minima and viewing the potential's dependence on the remaining two. This is shown in figures 2 – 5.

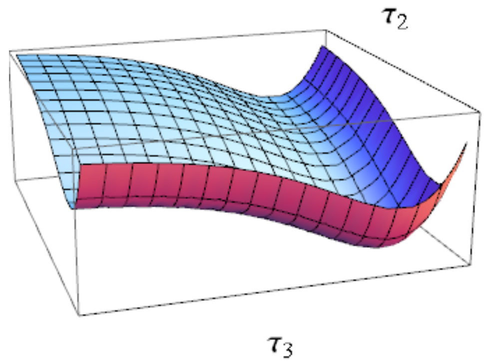

Figure 2: The potential with all but two τ_i and s,σ fixed.

To determine the field with the flattest approach, we should compute the slow roll parameters given in (6). We cannot rely on the plots as they have been done using somewhat arbitrary parameters. To do that we would have to differentiate the potential with respect to the canonically normalized fields. They are obtained by solving equation (11):

$$\begin{aligned} \tau_i^c &= \sqrt{4\lambda_i/3\mathcal{V}}\tau_i^{3/4} \\ s^c &= \tfrac{1}{\sqrt{2}}\log s \\ \sigma^c &= \sigma/\sqrt{2}s \end{aligned}$$

To extend the investigation, we could compute the slow roll parameters, the number of e-foldings and obtain a form for the spectral index; these can be readily checked against observational data. This would be relatively straightforward and would be helpful in providing constraints on the various parameters that appear in the effective theory. Further (and somewhat more involved) investigations could involve including the complex structure moduli in the potential and not considering them fixed.

[12] I would like to thank D. Grainger for clarifying this issue.

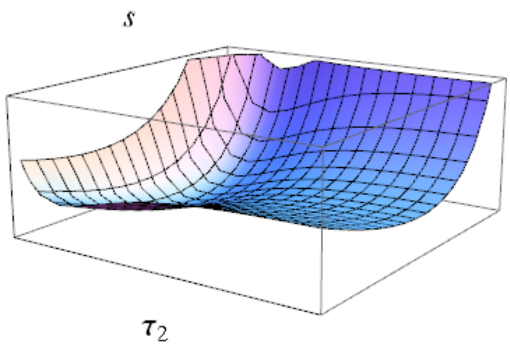

Figure 3: The potential with all but one τ_i and σ fixed.

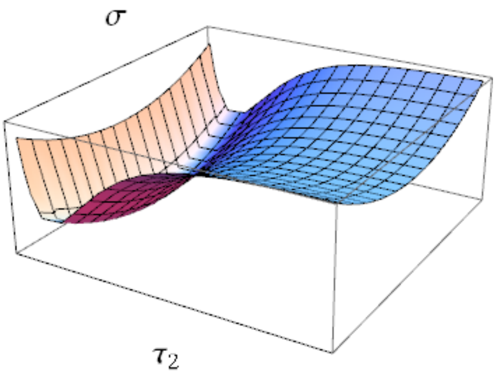

Figure 4: The potential with all but one τ_i and s fixed.

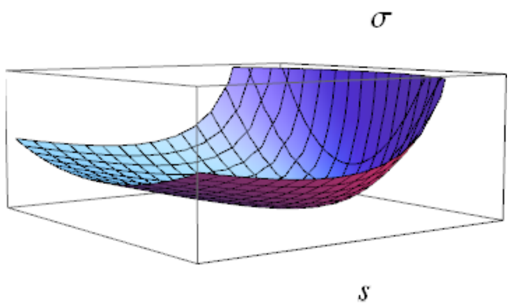

Figure 5: The potential with all τ_i fixed. This is very similar to the situation with just the dilaton.

8 Acknowledgments

I would like to thank Dr. Lukas for providing advice and assistance throughout the project. As mentioned in previous footnotes, I wish to thank D. Grainger and J. Conlon for clarifying some issues.

References

[1] Hobson, M.P., Efstathiou, G., Lasenby, A.N., 2006, *General Relativity: An Introduction for Physicists* (Cambridge University Press)

[2] Lyth, D.H., Riotto, A., 1999, *Particle Physics Models of Inflation and the Cosmological Density Perturbation*, arXiv:hep-ph/9807278v4

[3] Becker, K., Becker, M., Schwarz, J.H., 2006, *String Theory and M-Theory: A Modern Introduction* (Cambridge University Press)

[4] Stewart, E.D., 1995, *Inflation, supergravity, and superstrings*, Phys. Rev. D, **51**, 12

[5] Kiritsis, E., 2007, *String Theory in a Nutshell* (Princeton University Press)

[6] Bailin, D., Love, A., 1994 *Supersymmetric Gauge Field Theory and String Theory* (Taylor & Francis Group)

[7] Conlon, J.P., Quevedo, F., 2005, *Kähler Moduli Inflation*, arXiv:hep-th/0509012v2

[8] Balasubramanian, V., Berglund, P., Conlon, J.P., Quevedo, F., 2005, *Systematics of Moduli Stabilisation in Calabi-Yau Flux Compactifications*, arXiv:hep-th/0502058v2

[9] Becker, K., Becker, M., Haack, M., Louis, J., 2002, *Supersymmetry Breaking and α'-Corrections to Flux Induced Potentials*, arXiv:hep-th/0204254v3

A Equation of motion for scalar fields in supergravity

We have the Lagrangian density

$$\mathcal{L} = \sqrt{-g}\left[K_{a\bar{b}}\partial_\mu\phi^a\partial^\mu\bar{\phi}^{\bar{b}} - V(\phi^a,\bar{\phi}^{\bar{b}})\right]$$

and we wish to obtain the equation of motion for the ϕ^a. We use the Euler-Lagrange equations

$$\frac{\partial\mathcal{L}}{\partial\bar{\phi}^{\bar{c}}} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\phi}^{\bar{c}})} = 0$$

Evaluating each term, we obtain

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\bar{\phi}^{\bar{c}}} &= \sqrt{-g}\Big[(\partial_{\bar{c}}K_{a\bar{b}})\partial_\mu\phi^a\partial^\mu\bar{\phi}^{\bar{b}} \\
&\quad - \partial_{\bar{c}}V(\phi^a,\bar{\phi}^{\bar{b}})\Big] \\
\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\phi}^{\bar{c}})} &= \partial_\mu\left(\sqrt{-g}\left[K_{a\bar{c}}\partial^\mu\phi^a\right]\right) \\
&= \sqrt{-g}(\partial_\mu K_{a\bar{c}}\partial^\mu\phi^a) \\
&\quad + K_{a\bar{c}}\partial_\mu(\sqrt{-g}\partial^\mu\phi^a) \\
&= \sqrt{-g}\partial^\mu\phi^a(\partial_{\bar{b}}K_{a\bar{c}}\partial_\mu\phi^b \\
&\quad + \partial_{\bar{b}}K_{a\bar{c}}\partial_\mu\bar{\phi}^{\bar{b}}) \\
&\quad + K_{a\bar{c}}\partial_\mu(\sqrt{-g}\partial^\mu\phi^a).
\end{aligned}$$

The first step in the last equation uses the product rule, and the second stop uses the chain rule.

We note that since $K_{a\bar{b}} = \partial_a\partial_{\bar{b}}K$, then $\partial_{\bar{c}}K_{a\bar{b}} = \partial_{\bar{b}}K_{a\bar{c}}$ as partial derivatives always commute. Using this, we can cancel two terms when we substitute these terms into the Euler-Lagrange equation. We also note that

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\partial^\mu\phi^a) = \Box\phi$$

The equation of motion then becomes

$$K_{a\bar{c}}\Box\phi^a + \partial_b K_{a\bar{c}}\partial_\mu\phi^b\partial^\mu\phi^a + \partial_{\bar{c}}V = 0$$

We multiply by $K^{\bar{c}d}$ and define

$$\Gamma^a_{bc} = \frac{1}{2}K^{a\bar{d}}\partial_b K_{d\bar{c}} \Rightarrow \Gamma^d_{ab} = \frac{1}{2}K^{d\bar{c}}\partial_b K_{\bar{c}a}$$

in analogy with the Christoffel symbols from Riemann manifolds.

We then obtain the final form of the equation

$$\Box\phi^d + 2\Gamma^d_{ab}\partial_\mu\phi^a\partial^\mu\phi^b + K^{\bar{c}d}\partial_{\bar{c}}V = 0$$

B The potential due to Kähler moduli only

We have the following forms of the Kähler potential and the superpotential

$$\begin{aligned}
K &= -2\log\left[\alpha\left(\tau_i^{3/2} - \sum_{i=2}^{n}\lambda_i\tau_i^{3/2} + \frac{\xi}{2}\right)\right] \\
W &= W_0 + \sum_{i=2}^{n}A_i e^{-a_iT_i}
\end{aligned}$$

and we wish to calculate the potential as in eqn. (10).

We can calculate the following [13]

$$\begin{aligned}
K^{T_i\bar{T}_j}\partial_{\bar{T}_j}K &= -2\tau_i \\
\partial_{T_i}W &= -a_iA_ie^{-a_iT_i},\ (2\le i\le n) \\
K_{T_i\bar{T}_j} &= \frac{9\alpha^2\lambda_i\lambda_j\sqrt{\tau_i\tau_j}}{8(\mathcal{V}+\frac{\xi}{2})^2} - \frac{3\alpha\lambda_i\delta_{ij}}{8\sqrt{\tau_i}(\mathcal{V}+\frac{\xi}{2})}
\end{aligned}$$

where $\lambda_1 \equiv -1$. The matrix of second derivatives of K (which will be inverted to obtain $K^{i\bar{j}}$) is in general non-diagonal and as a result, non-trivial to invert.

However, in the expression for $\mathcal{V}$, $\tau_1^{3/2}$ is the dominant term [7], and the other 'τ_i's are small contributions. At large $\mathcal{V}$, then, $\mathcal{V}\sim\tau_1^{3/2}$. We list below $K_{i\bar{j}}$ with $i\neq j$.

$$\begin{aligned}
K_{T_1\bar{T}_1} &= \frac{3\alpha}{8\mathcal{V}}\left(\frac{3\alpha\sqrt{\tau_1}}{\mathcal{V}} - \frac{1}{\sqrt{\tau_1}}\right) \\
&= \mathcal{O}(\tau_1^{-5/2}) - \mathcal{O}(\tau_1^{-2}) \\
K_{T_i\bar{T}_i} &= \frac{3\alpha\lambda_i}{8\mathcal{V}}\left(\frac{3\alpha\lambda_i\sqrt{\tau_i}}{\mathcal{V}} - \frac{1}{\sqrt{\tau_i}}\right) \\
&= \mathcal{O}(\tau_i^{-1/2}\tau_1^{-3}) - \mathcal{O}(\tau_i^{-1/2}\tau_1^{-3/2}) \\
K_{T_1\bar{T}_i} &= \frac{-9\alpha^2\lambda_i\sqrt{\tau_1\tau_i}}{8\mathcal{V}^2} \\
&= \mathcal{O}(\tau_i^{1/2}\tau_1^{-5/2}) \\
K_{T_i\bar{T}_j} &= \frac{9\alpha^2\lambda_i\lambda_j\sqrt{\tau_i\tau_j}}{8\mathcal{V}^2} \\
&= \mathcal{O}(\tau_i^{1/2}\tau_i^{1/2}\tau_1^{-3})
\end{aligned}$$

If we now impose $\tau_i \gg \tau_i,\ i\neq 1$, we can see that we are quite justified in dropping all the

[13] I would like to thank J. Conlon for clarifying an issue regarding these results.

off-diagonal elements. Furthermore, we can ignore the first of the two terms for all the elements along the main diagonal. The form of the Kähler metric at large $\mathcal{V}$ then, is

$$K_{T_i\bar{T}_j} \simeq \frac{3\alpha\lambda_i}{8\mathcal{V}\sqrt{\tau_i}}\delta_{ij}$$

At large $\mathcal{V}$ we are also able to disregard the $\xi/2$ term [7], and this allows us to write $e^K \simeq 1/\mathcal{V}^2$. Inverting the matrix is now straightforward:

$$K^{T_i\bar{T}_j} \simeq \frac{8\mathcal{V}\sqrt{\tau_i}}{3\alpha\lambda_i}\delta_{ij}$$

The Kronecker delta drops the cross terms arising from $\partial_i W \partial_{\bar{j}} \bar{W}$, and using

$$\begin{aligned} w\bar{z} + z\bar{w} &= 2\mathfrak{Re}\{w\bar{z}\} \\ &= 2\mathfrak{Re}\{\bar{w}z\}, \ \forall\ w, z \in \mathbb{C} \end{aligned}$$

to simplify the $(W\partial_i K \partial_{\bar{j}} \bar{W} + c.c.)$ term, we can write the potential as (to leading order in $\mathcal{V}$)

$$V = \frac{1}{\mathcal{V}^2}\left[\sum_{i=2}^{n} \frac{8\mathcal{V}\sqrt{\tau_i}}{3\alpha\lambda_i} a_i^2 |A_i|^2 e^{-a_i(T_i+\bar{T}_i)} - \sum_{i=2}^{n} 2\tau_i a_i \times 2\mathfrak{Re}\{\bar{W}A_i e^{-a_i T_i}\}\right] + \frac{3\xi W_0^2}{4\mathcal{V}^3}$$

The sums only run from 2 to n since $\partial_{T_0} W = 0$. The appearance of the third term is described in Section 3 of [9]. Since $T_i + \bar{T}_i = 2\tau_i$, the imaginary part of T_i contributes only in the second term. We write $T_i = \tau_i + ib_i$ and $\bar{W}A_i = |W||A_i|e^{i\theta}$, where θ is the argument of $\bar{W}A_i$ to get

$$\begin{aligned} &\mathfrak{Re}\{\bar{W}A_i e^{-a_i T_i}\} \\ &= \mathfrak{Re}\{|W||A_i|e^{i\theta} e^{-a_i(\tau_i + ib_i)}\} \\ &= |W||A_i|e^{-a_i\tau_i}\mathfrak{Re}\{e^{i(\theta - ia_i b_i)}\} \\ &= |W||A_i|e^{-a_i\tau_i}\cos(\theta - ia_i b_i) \end{aligned}$$

Since we are looking for a minimum of the potential, we can minimize with respect to b_i which will multiply the term by -1 (as that is the minimum of any cosine function). We don't need to worry about the phases of W, A_i, as they don't appear in the potential anymore. We will write $W = |W|, A_i = |A_i|$ from here onwards. We are considering the limit of $\mathcal{V}$ as in [8], and so the nonperturbative corrections to W are small, and we replace it with W_0.

The final form then, is

$$\begin{aligned} V &= \frac{8}{3\mathcal{V}\alpha\lambda_i}\sum_{i=2}^{n}\frac{(a_i A_i e^{-a_i\tau_i})^2\sqrt{\tau_i}}{\lambda_i} \\ &\quad - \frac{4W_0}{\mathcal{V}^2}\sum_{i=2}^{n} a_i A_i \tau_i e^{-a_i\tau_i} + \frac{3\xi W_0^2}{4\mathcal{V}^3}. \end{aligned}$$

The derivatives are

$$\begin{aligned} \frac{\partial V}{\partial\tau_i} &= \frac{-4e^{-2a_i\tau_i}a_i A_i}{3\mathcal{V}^2\alpha\lambda_i\sqrt{\tau_i}}\Big(\mathcal{V}a_i A_i(4a_i\tau_i - 1) \\ &\quad - 3e^{a_i\tau_i}\alpha\lambda_i\sqrt{\tau_i}(a_i\tau_i - 1)\Big) \\ \frac{\partial^2 V}{\partial\tau_i\partial\tau_j} &= \frac{2e^{-2a_i\tau_i}a_i^2 A_i}{3\mathcal{V}^2\alpha\lambda_i\tau_i^{3/2}}\delta_{ij} \\ &\quad \times\Big(\mathcal{V}A_i(16a_i^2\tau_i^2 - 8a_i\tau_i - 1) \\ &\quad - 6e^{a_i\tau_i}\alpha W_0\lambda_i\tau_i^{3/2}(a_i\tau_i - 2)\Big) \end{aligned}$$

C The potential due to Kähler moduli and dilaton

We have the following forms of the Kähler potential and the superpotential

$$\begin{aligned} K &= -2\log\mathcal{V} - \log(S + \bar{S}) \\ W &= W_0 + \sum_{i=2}^{n} A_i e^{-a_i T_i} + c + dS \end{aligned}$$

and we wish to calculate the potential as in eqn. (10).

As before, we have $K^{T_i\bar{T}_j} \simeq 8\mathcal{V}\sqrt{\tau_i}\delta_{ij}/3\alpha\lambda_i$, $K^{S\bar{S}} = (2s)^{-2}$, and since K is a sum of the different terms, $K^{T_i\bar{S}} = 0$. We also have

$$\begin{aligned} K^{T_i\bar{T}_j}\partial_{T_i}K\partial_{\bar{T}_j}K &= 3 \\ K^{T_i\bar{T}_j}\partial_{\bar{T}_j}K &= -2\tau_i \\ K^{S\bar{S}}\partial_S K\partial_{\bar{S}}K &= 1 \\ K^{S\bar{S}}\partial_S K &= -2s \end{aligned}$$

Therefore, the terms arising from $K^{T_i\bar{T}_j}$ are identical as before. We obtain three new terms, however:

$$\begin{aligned} K^{S\bar{S}}(\partial_S W \partial_{\bar{S}} \bar{W}) &= 4s^2|c|^2 \\ K^{S\bar{S}}\partial_S K(\bar{W}\partial_S W + c.c.) &= -4s\mathfrak{Re}\{\bar{W}c\} \\ K^{S\bar{S}} W \partial_S K \bar{W} \partial_{\bar{S}} K &= |W|^2 \end{aligned}$$

We also add on $3\xi|W|^2/4\mathcal{V}^3$ for the same reasons as before, described in [9], although this time we cannot just take W_0, since the terms linear in S are not necessarily small.

Finally, we note that in the appropriate limit, $e^K \simeq 1/2\mathcal{V}^2 s$, and so we multiply across by that. This gives us

$$\begin{aligned} V = &\frac{4}{3\mathcal{V}\alpha s}\sum_{i=2}^{n}\frac{(a_iA_ie^{-a_i\tau_i})^2\sqrt{\tau_i}}{\lambda_i} \\ &+ \frac{2}{\mathcal{V}^2 s}\sum_{i=2}^{n} a_i\tau_i\mathfrak{Re}\{\bar{W}A_ie^{-a_iT_i}\} \\ &+ \frac{2}{\mathcal{V}^2}\left(s|c|^2 - \mathfrak{Re}\{\bar{W}c\}\right) + \frac{|W|^2}{2\mathcal{V}^2 s}\left(1+\frac{3\xi}{4\mathcal{V}}\right) \end{aligned}$$

As before, $\mathfrak{Im}\{T_i\} = b_i$ only appears in the second term, and we can make the same argument that minimizing with respect to that will only introduce an overall minus sign.

We introduce the same variables as in the main text: $\gamma \equiv 1 + 3\xi/4\mathcal{V}$, and $f \equiv \mathfrak{Re}\{\bar{W}b\}$. Note that γ is a constant with respect to the fields but f depends on s, σ through W. We compute the following:

$$\begin{aligned} \dot{W} \equiv \partial W/\partial s &= s|c|^2 + \mathfrak{Re}\{\bar{c}(W_0+d)\} \\ W' \equiv \partial W/\partial\sigma &= \sigma|c|^2 + \mathfrak{Im}\{\bar{c}(W_0+d)\} \\ \ddot{W} = W'' &= |c|^2 \\ \dot{W}' &= 0 \\ \dot{f} \equiv \partial f/\partial s &= |c|^2 \\ f' = \ddot{f} = f'' = \dot{f}' &= 0 \end{aligned}$$

The first and second derivatives are

$$\begin{aligned} \frac{\partial V}{\partial \tau_i} &= \frac{-4e^{-2a_i\tau_i}a_iA_i}{3\mathcal{V}^2\alpha\lambda_i\sqrt{\tau_i}}\Big(\mathcal{V}a_iA_i(4a_i\tau_i - 1) \\ &\quad - 3e^{a_i\tau_i}\alpha\lambda_i\sqrt{\tau_i}(a_i\tau_i - 1)\Big) \\ \frac{\partial V}{\partial\sigma} &= \frac{1}{\mathcal{V}^2}\Big(-2f' \\ &\quad + \frac{W'}{s}\Big(\gamma W - 2\sum_{i=2}^{n} a_iA_i\tau_ie^{-a_i\tau_i}\Big)\Big) \\ \frac{\partial V}{\partial s} &= \frac{1}{6\mathcal{V}^2}\Bigg[12|c|^2 - 12\dot{f} + \frac{1}{s^2} \\ &\quad \times\Big(-\frac{8\mathcal{V}}{\alpha}\sum_{i=2}^{n}\frac{(a_iA_ie^{-a_i\tau_i})^2\sqrt{\tau_i}}{\lambda_i} + 6sW\dot{W}\gamma \\ &\quad + 12(W - s\dot{W})\sum_{i=2}^{n} a_iA_i\tau_ie^{-a_i\tau_i} - 3\gamma W^2\Big)\Bigg] \\ \frac{\partial^2 V}{\partial\tau_i\partial\tau_j} &= \frac{e^{-2a_i\tau_i}a_i^2A_i}{3s\mathcal{V}^2\alpha\lambda_i\tau_i^{3/2}}\delta_{ij} \\ &\quad \times\Big(\mathcal{V}A_i(16a_i^2\tau_i^2 - 8a_i\tau_i - 1) \\ &\quad - 6e^{a_i\tau_i}\alpha W\lambda_i\tau_i^{3/2}(a_i\tau_i - 2)\Big) \\ \frac{\partial^2 V}{\partial\tau_i\partial\sigma} &= \frac{1}{s\mathcal{V}^2}\Big(2W'a_iA_ie^{-a_i\tau_i}(a_i\tau_i - 1)\Big) \\ \frac{\partial^2 V}{\partial\sigma^2} &= \frac{1}{s\mathcal{V}^2}\Big[\gamma W'^2 - 2sf'' \\ &\quad + W''\Big(\gamma W - 2\sum_{i=2}^{n} a_iA_i\tau_ie^{-a_i\tau_i}\Big)\Big] \\ \frac{\partial^2 V}{\partial s\partial\sigma} &= \frac{-1}{s^2\mathcal{V}^2}\Big(\gamma W'(W - s\dot{W}) \\ &\quad - 2\sum_{i=2}^{n} a_iA_i\tau_ie^{-a_i\tau_i}(W' - s\dot{W}') \\ &\quad - s\gamma W\dot{W}' + 2s^2\dot{f}'\Big) \\ \frac{\partial^2 V}{\partial\tau_i\partial s} &= \frac{2e^{-2a_i\tau_i}a_iA_i}{3s^2\mathcal{V}^2\alpha\lambda_i\sqrt{\tau_i}}\Big(\mathcal{V}a_iA_i(4a_i\tau_i - 1) \\ &\quad - 3e^{a_i\tau_i}\alpha\lambda_i\sqrt{\tau_i}(a_i\tau_i - 1)(W - s\dot{W})\Big) \\ \frac{\partial^2 V}{\partial s^2} &= \frac{1}{3s^3\mathcal{V}^2}\Bigg[\frac{8\mathcal{V}}{\alpha}\sum_{i=2}^{n}\frac{(a_iA_ie^{-a_i\tau_i})^2\sqrt{\tau_i}}{\lambda_i} \\ &\quad - 6(2W - 2s\dot{W} + s^2\ddot{W})\sum_{i=2}^{n} a_iA_i\tau_ie^{-a_i\tau_i} - 6s^3\ddot{f} \\ &\quad + 3\gamma(W^2 - 2sW\dot{W} + s^2\dot{W}^2 + s^2W\ddot{W})\Bigg] \end{aligned}$$

Most of these expressions simplify considerably when the derivatives of W and f are substituted.